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# Arbitrage and price revelation with private beliefs

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# Arbitrage and price revelation with private beliefs

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***Abstract***

*We extend the Cornet-de Boisdeffre (2002-2009) asymmetric information finite dimensional model to a more general setting, where agents may forecast prices with some private uncertainty. This new model drops both Radner's (1972-1979) classical, but restrictive, assumptions of rational expectations and perfect foresight. It deals with sequential financial equilibrium, when agents, unaware of how equilibrium prices or quantities are determined, are prone to uncertainty between - possibly uncountable - forecasts. Under perfect foresight, the extended model coincides with Cornet-de Boisdeffre's (2002-2009). Yet, when anticipations are private, we argue, any element of a typically uncountable 'minimum uncertainty set' may prevail as an equilibrium price tomorrow. This outcome is inconsistent with perfect foresight and appeals for a broader definition of sequential equilibrium, which we propose hereafter. By standard techniques, we embed and extend Cornet-de Boisdeffre's (2002-2009) results, to the infinite dimensional model. The aim is to lay foundations for another paper, showing that the concept of sequential equilibrium we propose may solve the classical existence problems of the perfect foresight model, following Hart (1974).*

**Key words**: sequential equilibrium, temporary equilibrium, perfect foresight, expectations, incomplete markets, asymmetric information, arbitrage.

**JEL Classification**: D52

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# 1 Introduction

In economies subject to uncertainty and asymmetric information, agents are traditionally assumed to have a 'model' of how equilibrium prices are determined, along Radner (1979), and to infer any additional information from comparing actual prices and price expectations to their theoretical values. This so-called 'rational expectations' hypothesis is convenient to deal with sequential equilibrium, but relies on a strong assumption regarding agents' capacities, and leads to standard cases of non-existence of equilibrium. In Cornet-de Boisdeffre (2002-2009) and de Boisdeffre (2007), we showed that treating asymmetric information without Radner's rational expectations assumption was always possible and improved existence results, namely, it guaranteed the existence of equilibrium on a purely financial market under the same conditions as for symmetric information.

Our former model of asymmetric information still retained the standard sequential equilibrium's second hypothesis, namely, Radner's (1972) perfect foresight, along which agents anticipate with certainty exactly one price for each commodity (or asset) in each random state, which turns out to be the true price, if that state prevails tomorrow. The rationale for doing this was technical. Indeed, keeping perfect foresight, while dropping rational expectations, might have been difficult to justify economically. Technically, however, perfect foresight is the easiest way to insure equilibrium unfolds sequentially, as opposed to '*temporarily*'. That is, when all uncertainty is removed, no agent would ever face bankruptcy or a welfare increasing trade opportunity, and agents' equilibrium decisions (ex ante) clear on markets at all time periods. These outcomes no longer hold at a temporary equilibrium, because agents may fail to forecast prices correctly at the outset.

Yet, perfect foresight is a sufficient but, by no means, a necessary condition to guarantee sequential unfolding. Though it has remained the standard setting so far, the perfect foresight model deals with a restrictive notion of sequential equilibrium, which embeds feasibility and existence problems, akin to the rational expectation model's dealing with asymmetric information. In particular, the perfect foresight model virtually requires the common knowledge of the equilibrium price, leads to standard existence problems, and barely explains speculation or crash phenomena, stemming from agents' beliefs. These shortcomings appeal for an alternative broader definition of the sequential financial equilibrium concept. The main purpose of this paper is to introduce such a concept and to lay foundations for proving its existence in a companion paper, by presenting, first and hereafter, a theory of arbitrage under asymmetric information in an infinite dimensional setting, where agents are prone to uncertainty between (possibly uncountable) price forecasts.

We do this by extending our earlier model of asymmetric information, so as to let agents be uncertain of future prices on each spot market, whenever required. We introduce new concepts of beliefs, structures of beliefs and refinements, no-arbitrage prices and the information they reveal. These concepts enhance those of our 2002 model to fit with the new setting, and coincide with the earlier ones under perfect foresight. Via standard, infinite dimensional analysis techniques, we extend the arbitrage properties of Cornet-de Boisdeffre (2002-2009) to the new model. In particular, we show how agents, with no price model a la Radner, may still update their beliefs from observing prices on current markets and, thereby, free markets from arbitrage opportunities. Along this inference process, agents' beliefs are said to be price-revealed. Besides the new results, this paper embeds all main properties demonstrated in the finite dimensional case by Cornet-de Boisdeffre (2002-2009) as a particular application - since the two models coincide under perfect foresight.

Formally, the model we propose is a two-period pure exchange economy, where finitely many agents face an exogenous uncertainty, represented by finitely many random states of nature (on which they may be asymmetrically informed), exchange goods on spot markets, for the purpose of consumption, and trade, unrestrained, on a (possibly incomplete) financial market, so as to transfer wealth across periods and states. At the first period, besides the above exogenous uncertainty on the future state of nature, agents may face an '*endogenous uncertainty*' on the future price, in each state they expect. Namely, consumers have private sets of anticipations for future spot prices, distributed along idiosyncratic probability laws, called beliefs. The latter uncertainty on prices is traditionally referred to as *'endogenous'*, because it concerns and may affect the endogenous variables.

The model's equilibrium, called '*correct foresight equilibrium'* (C.F.E.), is reached when agents anticipate tomorrow's '*true*' price as a possible outcome, and make optimal trade and consumption decisions today, given their preferences, which clear on markets and remain optimal ex post, given prior portfolio choices. In a companion paper, we show how a C.F.E. may be reached and why this equilibrium may solve the classical existence problems, which followed, not only Radner (1979), but Hart (1974). Whenever required, agents' revised beliefs at the C.F.E. may be revealed by the equilibrium price itself, in the sense defined above. This concept is, indeed, a *sequential* one, that is, differs from the *temporary* equilibrium's, introduced by Hicks (1939), developed by Grandmont (1977, 1982), Green (1973), Hammond (1983), Balasko (2003), among others, where agents' forecasts need not be correct.

The paper is organized as follows: we present the model, in Section 2, its arbitrage properties, in Section 3, and inference mechanisms, in Section 4.

# 2 The basic model

We consider a pure-exchange financial economy with two time-periods ($t \in \{0,1\}$) and two markets, a commodity market and a financial market. There is an a priori uncertainty at the first period ($t = 0$) about which state $s$ of a given state space $S$ will prevail at the second period ($t = 1$), when all uncertainty is removed. The state of nature at $t = 0$ is non random and denoted by $s = 0$. The sets of agents (or consumers), $I := \{1, ..., m\}$, of commodities, $\mathcal{L} := \{1, ..., L\}$, of states of nature, $S := \{1, ..., N\}$, and financial assets, $\mathcal{J} := \{1, ..., J\}$, are all finite subsets of $\mathbb{N}$.

Before presenting the model, we introduce notations, which are used throughout.

## 2.1 The model's notations

Throughout, we denote by $\cdot$ the scalar product and by $\|.\|$ the Euclidean norm on an Euclidean space, by $\mathcal{B}(K)$ the Borel sigma-algebra of a topological space, $K$. We let $s = 0$ be the non-random state at $t = 0$ and $S' := \{0\} \cup S$. For all sets $\Sigma \subset S'$ and tuples $(\varepsilon, s, l, x, x', y, y') \in \mathbb{R}_{++} \times \Sigma \times \mathcal{L} \times \mathbb{R}^{\Sigma} \times \mathbb{R}^{\Sigma} \times (\mathbb{R}^L)^{\Sigma} \times (\mathbb{R}^L)^{\Sigma}$, we denote by:

- $x_s \in \mathbb{R}$, $y_s \in \mathbb{R}^L$ the scalar and vector, indexed by $s \in \Sigma$, of (resp.) $x$ and $y$;
- $y_s^l$ the $l^{th}$ component of $y_s \in \mathbb{R}^L$;
- $x \leqslant x'$ and $y \leqslant y'$ (resp. $x << x'$ and $y << y'$) the relations $x_s \leqslant x'_s$ and $y_s^l \leqslant y_s'^l$ (resp. $x_s < x'_s$ and $y_s^l < y_s'^l$) for all $(l, s) \in \{1, ..., L\} \times \Sigma$;
- $x < x'$ (resp. $y < y'$) the joint relations $x \leqslant x'$, $x \neq x'$ (resp. $y \leqslant y'$, $y \neq y'$);
- $\mathbb{R}^{L\Sigma} := (\mathbb{R}^L)^{\Sigma}$, $\mathbb{R}_+^{L\Sigma} = \{x \in \mathbb{R}^{L\Sigma} : x \geqslant 0\}$ and $\mathbb{R}_+^{\Sigma} := \{x \in \mathbb{R}^{\Sigma} : x \geqslant 0\}$, $\mathbb{R}_{++}^{L\Sigma} := \{x \in \mathbb{R}^{L\Sigma} : x >> 0\}$ and $\mathbb{R}_{++}^{\Sigma} := \{x \in \mathbb{R}^{\Sigma} : x >> 0\}$,
- $\mathcal{M}_0 := \{(p_0, q) \in \mathbb{R}_+^L \times \mathbb{R}^J : \|p_0\| + \|q\| = 1\}$;

- $\mathcal{M}_s := \{(s, p_s) : p_s \in \mathbb{R}_+^L, \ \|p_s\| = 1\}$ and $\mathcal{M}_s^\varepsilon := \{(s, p_s) \in \mathcal{M}_s : p_s \in [\varepsilon, 1]^L\}$, for $s \in S$;
- $\mathcal{M} := \cup_{s \in S} \mathcal{M}_s$ and $\mathcal{M}^\varepsilon := \cup_{s \in S} \mathcal{M}_s^\varepsilon$.

### 2.2 The commodity and asset markets

The $L$ commodities, $l \in \mathcal{L}$, are used for the purpose of consumption and may be exchanged between agents on spot markets. There are $\#S'$ ex ante possible spot markets, namely one in each state $s \in S'$. In each state $s \in S$, an expectation of a spot price, $p \in \mathbb{R}_+^L$, is denoted by the pair $\omega_s := (s, p) \in S \times \mathbb{R}_+^L$ (which will also stand for the spot price $p$ in state $s$ itself). At little cost, we normalize admissible expectations and spot prices, in each state $s \in S$, to the above defined set $\mathcal{M}_s$.

Agents exchange commodities in order to increase their welfare. Ex post, the generic $i^{th}$ agent's welfare is measured by $u_i(x, y) \in \mathbb{R}_+$, where $x := (x^1, ..., x^L) \in \mathbb{R}_+^L$ and $y := (y^1, ..., y^L) \in \mathbb{R}_+^L$ are the vectors of consumptions, respectively, at $t = 0$ and $t = 1$, and $u_i : \mathbb{R}_+^{2L} \to \mathbb{R}_+$ is a utility function, assumed to be continuous.

Trade may take place because each agent, $i \in I$, can rely on an endowment, $e_i := (e_{is}) \in \mathbb{R}_{++}^{LS'}$, of the $L$ goods, which grants her the commodity bundle $e_{i0} \in \mathbb{R}_{++}^L$ at $t = 0$, and $e_{is} \in \mathbb{R}_{++}^L$, in each state $s \in S$ if this state prevails at $t = 1$. For the sake of simpler notations, we henceforth let $e_{i\omega} := e_{is}$, for every triple $(i, s, \omega) \in I \times S' \times \mathcal{M}_s$.

The financial market permits limited transfers across periods and states, via $J$ assets, also called securities, $j \in \mathcal{J} := \{1, ..., J\}$, which are exchanged at $t = 0$ and pay off at $t = 1$. Assets may be nominal or real (i.e., pay off in account units or in commodities). For any spot price, or expectation, $\omega \in \mathcal{M}$, the payoffs, $v_j(\omega) \in \mathbb{R}$, of each asset $j \in \{1, ..., J\}$ conditional on the occurence of $\omega$, define a row vector,

$V(\omega) = (v_j(\omega)) \in \mathbb{R}^J$, and the mapping $\omega \in \mathcal{M} \mapsto V(\omega)$ is continuous from the definition and the continuity of the scalar product (since assets pay in money or commodities).

Provided she can afford, every agent $i \in I$ may take unrestrained positions, $z_i^j \in \mathbb{R}$ (positive, if purchased; negative, if sold), in every security $j \in \{1, ..., J\}$, which define her portfolio, $z_i := (z_i^j) \in \mathbb{R}^J$. When an asset price, $q \in \mathbb{R}^J$, is observed at $t = 0$, a portofolio, $z \in \mathbb{R}^J$, is thus a contract, which costs $q \cdot z$ units of account at $t = 0$, and promises to pay $V(\omega) \cdot z$ units tomorrow, for each spot price $\omega \in \mathcal{M}$, if $\omega$ obtains. Similarly, we henceforth normalize first period prices, $\omega_0 := (p_0, q)$, to the set $\mathcal{M}_0$.

### 2.3 Information and beliefs

During the first period ($t = 0$), each agent receives a private information signal, $S_i \subset S$, which informs her that the true state, which will prevail at $t = 1$, will be in $S_i$. Henceforth, the collection $(S_i)$ of all agents' signals is set as given and we let $\underline{\mathbf{S}} := \cap_{i=1}^m S_i$, referred to as the pooled information set. Agents are assumed to receive no wrong information in the sense that no state of $S \backslash \underline{\mathbf{S}}$ will prevail, hence, $\underline{\mathbf{S}}$ is non-empty. Agents form private anticipations of future spot prices in each state they expect, distributed along idiosyncratic probability laws. Formally:

**Definition 1** *For all probability $\pi$, on $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$, and pair $(\omega := (s,p), \varepsilon) \in \mathcal{M} \times \mathbb{R}_{++}$, we let $B(\omega, \varepsilon) := \{(s', p') \in \mathcal{M} : \|p' - p\| + |s' - s| < \varepsilon\}$ and $P(\pi) := \{\omega \in \mathcal{M} : \pi(B(\omega, \varepsilon)) > 0, \forall \varepsilon > 0\}$ be a compact set, whose elements are called anticipations, expectations or forecasts. A probability, $\pi$, on $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$, is called a belief if the following Condition holds:*

$(a)$ $\exists \varepsilon \in \mathbb{R}_{++} : P(\pi) \subset \mathcal{M}^\varepsilon$.

*We denote by $\mathcal{B}$ the set of all beliefs. A belief $\pi\prime \in \mathcal{B}$ is said to refine $\pi \in \mathcal{B}$ and we denote it by $\pi\prime \leq \pi$, if the following Condition holds:*

$(b)$ $P(\pi') \subset P(\pi)$.

*Two beliefs,* $(\pi, \pi\prime) \in \mathcal{B}^2$*, are said to be equivalent, and we denote it by* $\pi\prime \sim \pi$*, if both relations* $\pi\prime \leq \pi$ *and* $\pi \leq \pi\prime$ *hold, and we let* $\overset{o}{\pi} := \{\overline{\pi} \in \mathcal{B} : \overline{\pi} \sim \pi\}$ *be their equivalence class. We denote by* $\mathcal{CB} := \{\overset{o}{\overline{\pi}} : \overline{\pi} \in \mathcal{B}\}$ *the set of classes, forming a partition, of* $\mathcal{B}$*, and by* $P(\overset{o}{\pi})$ *the expectation support of any class* $\overset{o}{\pi} \in \mathcal{CB}$*, namely, the set of anticipations,* $P(\overset{o}{\pi}) := P(\overline{\pi})$*, which is common to all beliefs* $\overline{\pi} \in \overset{o}{\pi}$*, and which characterizes* $\overset{o}{\pi}$*. We say that a class,* $\overset{o}{\pi}\prime \in \mathcal{CB}$*, refines* $\overset{o}{\pi} \in \mathcal{CB}$*, and denote it by* $\overset{o}{\pi}\prime \leq \overset{o}{\pi}$*, if* $P(\overset{o}{\pi}\prime) \subset P(\overset{o}{\pi})$*.*

*A collection of beliefs,* $(\pi_i) \in \mathcal{B}^m$*, is called a structure (of beliefs), and we denote it by* $(\pi_i) \in \mathcal{SB}$*, if the following Condition holds:*

*(c)* $\cap_{i=1}^m P(\pi_i) \neq \varnothing$*.*

*Similarly, a collection of classes,* $(\overset{o}{\pi}_i) \in \mathcal{CB}^m$*, is called a class structure (of beliefs), and we denote it by* $(\overset{o}{\pi}_i) \in \mathcal{CSB}$*, if* $\cap_{i=1}^m P(\overset{o}{\pi}_i) \neq \varnothing$*.*

*Let* $((\pi_i), (\pi'_i)) \in \mathcal{SB}^2$*,* $(\overset{o}{\pi}_i) \in \mathcal{CSB}$ *and payoff mapping,* $V$*, be given. The couples,* $[V, (\pi_i)]$ *and* $[V, (\overset{o}{\pi}_i)]$*, are called, respectively, a structure and a class structure (of payoffs and beliefs). The structure* $(\pi'_i)$ *is said to refine* $(\pi_i)$*, and we denote it by* $(\pi'_i) \leq (\pi_i)$*, if the relations* $\pi'_i \leq \pi_i$ *hold for each* $i \in I$*. The two structures are equivalent, and we denote it by* $(\pi_i) \sim (\pi'_i)$*, if both relations* $(\pi_i) \leq (\pi'_i)$ *and* $(\pi'_i) \leq (\pi_i)$ *hold. A refinement,* $(\pi^*_i) \in \mathcal{SB}$*, of* $(\pi_i) \in \mathcal{SB}$*, is said to be self-attainable if the following Condition holds:*

*(d)* $\cap_{i=1}^m P(\pi^*_i) = \cap_{i=1}^m P(\pi_i)$*.*

*The notions of refinement and self-attainable refinement are defined alike on* $\mathcal{CSB}$*.*

*Remark 1* Without changing the paper's results, a belief could be defined as a probability on $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$, whose support cannot take arbitrary large or low values. With normalized expectations this is stated by Condition $(a)$. Under perfect foresight, class structures, $(\overset{o}{\pi}_i) \in \mathcal{CSB}$, and information structures, $(S_i)$, coincide, as well as the above definitions of refinements with Cornet-de Boisdeffre's (2002).

## 2.4 Consumers' behavior and the notion of equilibrium

In this sub-Section, we assume that agents make their trade and consumption plans after having reached a (final) structure of beliefs, $(\pi_i) \in \mathcal{SB}$, and observed the market price at $t = 0$, $\omega_0 := (p_0, q) \in \mathcal{M}_0$, which are set as given and referred to throughout. The generic $i^{th}$ agent's consumption set is, then, defined as:

$$\boxed{X(\pi_i) := \mathcal{C}(\{0\} \cup P(\pi_i), \mathbb{R}^L_+)},$$

where $\mathcal{C}(\{0\} \cup P(\pi_i), \mathbb{R}^L_+)$ stands for the set of continuous mappings from $\{0\} \cup P(\pi_i)$ to $\mathbb{R}^L_+$. A consumption, $x \in X(\pi_i)$, is, thus, a mapping, which relates continuously $s = 0$ to a (fixed) consumption decision, $x_0 := x_{\omega_0} \in \mathbb{R}^L_+$, at $t = 0$, and every anticipation, $\omega := (s, p) \in P(\pi_i)$, to a random consumption decision $x_\omega \in \mathbb{R}^L_+$ at $t = 1$, which is conditional on the occurence of the spot price $\omega$, that is, of both state $s \in S$ and price $p \in \mathbb{R}^L_+$, on the spot market, at $t = 1$.

Each agent $i \in I$ elects and implements a consumption and investment decision, or strategy, $(x, z) \in X(\pi_i) \times \mathbb{R}^J$, that she can afford on markets, given her endowment, $e_i \in \mathbb{R}^{LS'}_+$, and her expectation set, $P(\pi_i)$. This defines her budget set as follows:

$$\boxed{B_i(\omega_0, \pi_i) := \{(x, z) \in X(\pi_i) \times \mathbb{R}^J : p_0 \cdot (x_0 \text{-} e_{i0}) \leqslant -q \cdot z; \;\; p_s \cdot (x_\omega \text{-} e_{i\omega}) \leqslant V(\omega) \cdot z, \forall \omega := (s, p_s) \in P(\pi_i)\}}.$$

An allocation, $(x_i) \in X[(\pi_i)] := \Pi_{i=1}^m X(\pi_i)$, is a collection of consumptions across consumers. We define the following sets of attainable allocations, for every price collection, $(\omega_s) := (\omega_s)_{s \in \underline{\mathbf{S}}} \in \Pi_{s \in \underline{\mathbf{S}}} \mathcal{M}_s$, and attainable portfolios, respectively:

$$\mathcal{A}((\omega_s), (\pi_i)) := \{(x_i) \in X[(\pi_i)] : \textstyle\sum_{i=1}^m (x_{i0} \text{-} e_{i0}) = 0, \; \sum_{i=1}^m (x_{i\omega_s} \text{-} e_{i\omega_s}) = 0, \; \forall s \in \underline{\mathbf{S}}, \; s.t. \; \omega_s \in \cap_{i=1}^m P(\pi_i)\} \;;$$

$$\mathcal{Z} := \{(z_i) \in (\mathbb{R}^J)^m : \textstyle\sum_{i=1}^m z_i = 0\}.$$

Hence, only spot markets, whose price is commonly expected (or observed at $t=0$) by all agents, need clear, along the above definition of an attainable allocation.

Each agent $i \in I$ has preferences represented by the V.N.M. utility function:

$$\boxed{u_i^{\pi_i} : x \in X(\pi_i) \mapsto u_i^{\pi_i}(x) := \int_{\omega \in P(\pi_i)} u_i(x_0, x_\omega) d\pi_i(\omega)}.$$

The generic $i^{th}$ agent's behavior is, then, to elect a strategy, which maximises this utility function in the buget set, that is, a strategy in $B_i^*(\omega_0, \pi_i) := \arg\max_{(x,z) \in B_i(\omega_0, \pi_i)} \ u_i^{\pi_i}(x)$.

The above economy is denoted by $\mathcal{E}$, whose equilibrium is defined as follows:

**Definition 2** *A collection of prices,* $(\omega_s) \in \Pi_{s \in \underline{\mathbf{S}}'} \mathcal{M}_s$*, beliefs,* $(\pi_i) \in \mathcal{SB}$*, and strategies,* $(x_i, z_i) \in B_i(\omega_0, \pi_i)$*, defined for each* $i \in I$*, is a sequential equilibrium (respectively, a temporary equilibrium) of the economy* $\mathcal{E}$*, or correct foresight equilibrium (C.F.E.), if the following Conditions* $(a)$-$(b)$-$(c)$-$(d)$ *(resp., Conditions* $(b)$-$(c)$-$(d)$*) hold:*

$(a)$ $\forall s \in \underline{\mathbf{S}},\ \omega_s \in \cap_{i=1}^m P(\pi_i)$*;*

$(b)$ $\forall i \in I,\ (x_i, z_i) \in B_i^*(\omega_0, \pi_i) := \arg\max_{(x,z) \in B_i(\omega_0, \pi_i)} \ u_i^{\pi_i}(x)$*;*

$(c)$ $(x_i) \in \mathcal{A}((\omega_s), (\pi_i))$*;*

$(d)$ $(z_i) \in \mathcal{Z}$*.*

*Remark 2* Along Definition 2, a C.F.E. is reached when agents forecast prices correctly (Condition $(a)$), make optimal decisions at $t=0$ (Condition $(b)$), which clear on all markets at all dates (Conditions $(c)$-$(d)$), and is, indeed, a sequential equilibrium. At $t=1$, agents never face bankruptcy and have no incentive to exchange on the spot market. Under perfect foresight, the above sequential equilibrium concept coincides with Cornet-de Boisdeffre's (2002, Definition 2.3, p. 399).

# 3 The arbitrage properties

We define and characterize no-arbitrage prices and the information they reveal.

## 3.1 The model's no-arbitrage prices

We start with a standard application of separation theorems in topological vector spaces, which yields a no-arbitrage characterization used throughout the paper.

**Claim 1** *Let $\pi \in \mathcal{B}$ and $q \in \mathbb{R}^J$ be given. The following statements are equivalent:*

*$(i)$ there is no portfolio $z \in \mathbb{R}^J$, such that $-q \cdot z \geqslant 0$ and $V(\omega) \cdot z \geqslant 0$ for every $\omega \in P(\pi)$, with at least one strict inequality*;

*$(ii)$ there exists a continuous mapping $\lambda : P(\pi) \to \mathbb{R}_{++}$, such that $q = \int_{\omega \in P(\pi)} \lambda(\omega) V(\omega) d\pi(\omega)$.*

*Remark 3* It follows from Claim 1, that if a belief, $\pi \in \mathcal{B}$, meets the conditions of Claim 1 for a given price, $q \in \mathbb{R}^J$, any other equivalent belief, $\pi' \sim \pi$, meets the same conditions. This will make Definition 3, below, consistent.

**Proof** Let $\pi \in \mathcal{B}$ and $q \in \mathbb{R}^J$ be given.

$(ii) \Rightarrow (i)$ Assume that assertion $(ii)$ holds, and, let $z \in \mathbb{R}^J$ be given such that $-q \cdot z \geqslant 0$ and $V(\omega) \cdot z \geqslant 0$ for every $\omega \in P(\pi)$. Assume, first, that $V(\overline{\omega}) \cdot z > 0$, for some $\overline{\omega} \in P(\pi)$. Then, the above inequalities $V(\omega) \cdot z \geqslant 0$, which hold every $\omega \in P(\pi)$, and the continuity of $V$ at $\overline{\omega} \in P(\pi)$ imply $q \cdot z = \int_{\omega \in P(\pi)} \lambda(\omega) V(\omega) \cdot z d\pi(\omega) > 0$, in contradiction with the above relation -$q \cdot z \geqslant 0$. Hence, $V(\omega) \cdot z = 0$, for all $\omega \in P(\pi)$, which also yields, from Assertion $(ii)$, $q \cdot z = \int_{\omega \in P(\pi)} \lambda(\omega) V(\omega) \cdot z d\pi(\omega) = 0$, that is, assertion $(i)$ holds. $\square$

$(i) \Rightarrow (ii)$ Assume that Assertion $(i)$ holds and let $P := \{s = 0\} \cup P(\pi)$ and $\mathcal{C}(P, \mathbb{R})$ be the set of continuous (hence, Borel measurable) mappings from $P$ to $\mathbb{R}$, endowed with the (well defined) operator $(f, g) \in \mathcal{C}(P, \mathbb{R})^2 \mapsto < f, g > := f(0)g(0) + \int_{\omega \in P(\pi)} f(\omega) g(\omega) d\pi(\omega)$, the norm $f \in \mathcal{C}(P,\mathbb{R}) \mapsto \|f\| := \sqrt{f(0)^2 + \int_{\omega \in P(\pi)} f(\omega)^2 d\pi(\omega)}$, the induced metric and topology. The set $\mathcal{C}(P,\mathbb{R})$ is a convex metric space, with the linear sub-spaces:

$$A := \{f \in \mathcal{C}(P,\mathbb{R}) : \exists z \in \mathbb{R}^J,\ f(0) = -q \cdot z \ and\ f(\omega) = V(\omega) \cdot z,\ \forall \omega \in P(\pi)\};$$

$$A^{\perp} := \{f \in \mathcal{C}(P,\mathbb{R}) : < a, f > = 0,\ \forall a \in A\}.$$

Let $\mathcal{C}(P,\mathbb{R}_+)$ and $\mathcal{C}(P,\mathbb{R}_{++})$ be, respectively, the subsets of continuous non-negative and strictly positive valued mappings of $\mathcal{C}(P,\mathbb{R})$. Assertion $(i)$ is written $A \cap \mathcal{C}(P,\mathbb{R}_+) = \{0\}$. Assume, by contraposition, that $A^{\perp} \cap\ \mathcal{C}(P,\mathbb{R}_{++}) = \varnothing$, i.e., Assertion $(ii)$ fails.

Then, from the Interior Separating Hyperplane Theorem and the fact that $A^{\perp}$ is a linear sub-space (see Aliprantis-Border (1999), pp. 188, 190), there exists a nonzero continuous linear functional, $\varphi$, which properly separates $A^{\perp}$ and $\mathcal{C}(P,\mathbb{R}_{++})$, and such that $\varphi(a) = 0 \leqslant \varphi(b)$, for every $(a,b) \in A^{\perp} \times \mathcal{C}(P,\mathbb{R}_{++})$.

From Riesz' Theorem (see Aliprantis-Border (1999), p. 440), there exists $f \in \mathcal{C}(P,\mathbb{R})$, such that $\varphi(h) = < f, h >$, for every $h \in \mathcal{C}(P,\mathbb{R})$. The linear space $A$ is closed hence, with obvious definition, $A^{\perp\perp} = A$ (Aliprantis-Border (1999), p. 215). Then, from the above inequalities, the relations $f \in A^{\perp\perp} \cap \mathcal{C}(P,\mathbb{R}_+)\backslash\{0\} = A \cap \mathcal{C}(P,\mathbb{R}_+)\backslash\{0\}$ hold and contradict the above formulation, $A \cap\ \mathcal{C}(P,\mathbb{R}_+) = \{0\}$, of assertion $(i)$. □

We can now define and characterize arbitrage-free prices, beliefs, and structures.

**Definition 3** *Let a class structure of payoffs and beliefs,* $[V, (\overset{o}{\pi}_i)]$*, a class of beliefs,* $\overset{o}{\pi} \in \mathcal{CB}$*, a representative belief,* $\pi \in \overset{o}{\pi}$*, and a price,* $q \in \mathbb{R}^J$*, be given. The couples,* $(V, \overset{o}{\pi})$ *or* $(V, \pi)$*, are said to to be* $q$*-arbitrage-free (hence, arbitrage-free), or* $q$ *to be a no-arbitrage price of* $(V, \overset{o}{\pi})$*, or* $(V, \pi)$*, if the following equivalent Conditions hold:*
$(a)$ *there is no portfolio* $z \in \mathbb{R}^J$*, such that* $-q \cdot z \geqslant 0$ *and* $V(\omega) \cdot z \geqslant 0$ *for every* $\omega \in P(\pi) = P(\overset{o}{\pi})$*, with at least one strict inequality;*

*(b) there exists a continuous mapping* $\lambda : P(\pi) \to \mathbb{R}_{++}$*, such that* $q = \int_{\omega \in P(\pi)} \lambda(\omega)V(\omega)d\pi(\omega)$.

*We let* $Q[V, \overset{o}{\pi}]$ *be the set of no-arbitrage prices of* $(V, \overset{o}{\pi})$ *(or* $(V, \pi)$*) and* $Q_c[V, (\overset{o}{\pi}_i)] := \cap_{i=1}^{m} Q[V, \overset{o}{\pi}_i]$ *be the set of common no-arbitrage prices of* $[V, (\overset{o}{\pi}_i)]$*. The class structure* $[V, (\overset{o}{\pi}_i)]$ *is said to be arbitrage-free (resp.,* $q$*-arbitrage-free) if* $Q_c[V, (\overset{o}{\pi}_i)] \neq \varnothing$ *(resp., if* $q \in Q_c[V, (\overset{o}{\pi}_i)]$*). We say that* $q$ *is a no-arbitrage price (resp., a self-attainable no-arbitrage price) of* $[V, (\overset{o}{\pi}_i)]$ *if there exists a refinement (resp., a self-attainable refinement),* $(\overset{o}{\pi}{}_i^*) \leq (\overset{o}{\pi}_i)$*, such that* $q \in Q_c[V, (\overset{o}{\pi}{}_i^*)]$*, and we denote their set by* $Q[V, (\overset{o}{\pi}_i)]$*.*

*All above definitions and notations can be stated, equivalently from Remark 3, in terms of any representative structure,* $(\pi_i) \in \Pi_{i=1}^{m} \overset{o}{\pi}_i$*. We then refer to* $Q_c[V, (\pi_i)] := Q_c[V, (\overset{o}{\pi}_i)]$ *and* $Q[V, (\pi_i)] := Q[V, (\overset{o}{\pi}_i)]$ *as, respectively, the sets of common no-arbitrage prices, and no-arbitrage prices, of the structure* $[V, (\pi_i)]$*. When no confusion is possible, the reference to* $V$ *may be dropped in all above definitions and notations.*

*Remark 4* We notice that a symmetric refinement of any structure $(\pi_i) \in \mathcal{SB}$, that is, a refinement $(\pi'_i) \leq (\pi_i)$, such that $P(\pi'_i) = P(\pi'_1)$, for every $i \in I$, is always arbitrage-free along Definition 3. The latter Definition embeds and extends Cornet-de Boisdeffre's (2002) Definition 2.2 (p. 397). Henceforth, without recalling, we let the reader notice the same conclusion for all subsequent definitions and properties.

The following no-arbitrage characterization will be used throughout the paper.

**Claim 2** *Given* $(\overset{o}{\pi}_i) \in \mathcal{CSB}$*,* $(\pi_i) \in \Pi_{i=1}^{m} \overset{o}{\pi}_i$ *and* $q \in \mathbb{R}^J$*, the following Conditions are equivalent, the latter being called absence of future arbitrage opportunity (AFAO):*

$(i)$ the class structure $(\overset{o}{\pi}_i)$ *(or, equivalently, the structure* $(\pi_i)$*) is arbitrage-free*;

*(ii) there is no portfolio collection* $(z_i) \in (\mathbb{R}^J)^I$*, such that* $\sum_{i=1}^{m} z_i = 0$ *and* $V(\omega_i) \cdot z_i \geqslant 0$ *for every couple* $(i, \omega_i) \in I \times P(\pi_i) = I \times P(\overset{o}{\pi}_i)$*, with at least one strict inequality.*

**Proof** Let $(\overset{o}{\pi}_i) \in \mathcal{CSB}$ and $(\pi_i) \in \Pi_{i=1}^m \overset{o}{\pi}_i$ be given.

$(i) \Rightarrow (ii)$ Assume that $(\pi_i) \in \mathcal{SB}$ is arbitrage-free. Set as given $q \in Q_c[V, (\pi_i)]$, and, for each $i \in I$, a continuous mapping, $\lambda_i : P(\pi_i) \to \mathbb{R}_{++}$, such that $q = \int_{\omega \in P(\pi_i)} \lambda_i(\omega) V(\omega) d\pi_i(\omega)$, which exist from Definition 3. Assume that there exists $(z_i) \in (\mathbb{R}^J)^I$, such that $V(\omega_i) \cdot z_i \geqslant 0$ for every $(i, \omega) \in I \times P(\pi_i)$, with at least one strict inequality. Then, summing up the latter relations, wheigthed by $\lambda_i(\omega)$, for every $(i, \omega) \in I \times P(\pi_i)$, yields, from above and the continuity of $V$, $\sum_{i=1}^m q \cdot z_i = \sum_{i=1}^m \int_{\omega \in P(\pi_i)} \lambda_i(\omega) V(\omega) \cdot z_i d\pi_i(\omega) > 0$, which implies that $\sum_{i=1}^m z_i \neq 0$ and, consequently, that Condition AFAO holds. $\square$

$(ii) \Rightarrow (i)$ Assume Condition AFAO holds. The proof is akin to that of Claim 1.

For each $i \in I$, we let $P_i := \{s = 0\} \cup P(\pi_i)$, and $\mathcal{C}(P_i, \mathbb{R})$ be the set of continuous mappings from $P_i$ to $\mathbb{R}$, endowed with the operator, $(f, g) \in \mathcal{C}(P_i, \mathbb{R})^2 \mapsto < f, g > := f(0)g(0) + \int_{\omega \in P(\pi_i)} f(\omega) g(\omega) d\pi_i(\omega)$, and the induced norm, metric and topology, as in Claim 1. Then, we endow from above the set $\mathcal{C} := \Pi_{i=1}^m \mathcal{C}(P_i, \mathbb{R})$ with the operator, metric and topology of product spaces, let $\mathcal{C}_+$ and $\mathcal{C}_{++}$ be the subsets of non-negative and strictly positive valued functions of $\mathcal{C}$ and $A$, $A^\perp$ be the linear sub-spaces:

$$A := \{(f_i) \in \mathcal{C} : (f_i(0)) = 0,\ \exists (z_i) \in \mathbb{R}^{JI} : \textstyle\sum_{i=1}^m z_i = 0,\ f_i(\omega_i) = V(\omega_i) \cdot z_i,\ \forall (i, \omega_i) \in I \times P(\pi_i)\};$$
$$A^\perp := \{f \in \mathcal{C} : < a, f > = 0, \forall a \in A\}.$$

The AFAO Condition is written: $A \cap \mathcal{C}_+ = \{0\}$. If $A^\perp \cap \mathcal{C}_{++} = \varnothing$, the very same arguments as in Claim 1 apply, and, as we let the reader check, yield a contradiction. Hence, we may set as given $(\lambda_i) \in A^\perp \cap \mathcal{C}_{++} \neq \varnothing$. Then, by taking $(z_i) \in (\mathbb{R}^J)^I$, such that $(z_i, z_j) = (-z_1, 0)$, for every $(i, j) \in I^2$, $i \neq 1$, $j \notin \{1, i\}$, the relation $(\lambda_i) \in A^\perp$ yields: $\int_{\omega \in P(\pi_i)} \lambda_i(\omega) V(\omega) \cdot z d\pi_i(\omega) = \int_{\omega \in P(\pi_1)} \lambda_1(\omega) V(\omega) \cdot z d\pi_1(\omega)$, for every pair $(i, z) \in I \times \mathbb{R}^J$.

Let $q := \int_{\omega \in P(\pi_1)} \lambda_1(\omega) V(\omega) d\pi_1(\omega)$. From above, $q = \int_{\omega \in P(\pi_i)} \lambda_i(\omega) V(\omega) d\pi_i(\omega)$, for every $i \in I$, which implies, from Definition 3, that $q \in Q_c[V, (\pi_i)]$, i.e., assertion $(i)$ holds. $\square$

### 3.2 Individual beliefs revealed by prices

First, we show a structure of beliefs admits a coarsest arbitrage-free refinement.

**Claim 3** *Let a class structure of payoffs and beliefs,* $[(V, (\overset{o}{\pi}_i)]$, *be given. Then, there exists a unique coarsest arbitrage-free refinement of* $(\overset{o}{\pi}_i) \in \mathcal{CSB}$, *namely, a refinement,* $(\overset{o}{\pi}{}_i^*) \leq (\overset{o}{\pi}_i)$, *which satisfies the two following Conditions:*

*(i)* $[V, (\overset{o}{\pi}{}_i^*)]$ *is arbitrage-free;*

*(ii) every arbitrage-free refinement of* $(\overset{o}{\pi}_i)$ *is a refinement of* $(\overset{o}{\pi}{}_i^*)$.

*The coarsest arbitrage-free refinement, denoted* $\overset{o}{\Pi}[V, (\overset{o}{\pi}_i)]$ *or* $\overset{o}{\Pi}[(\overset{o}{\pi}_i)]$, *is self-attainable.*

**Proof** Along Definitions 1 & 3, let $(\overset{o}{\pi}_i) \in \mathcal{CSB}$ be a given class structure, and let $\mathcal{R}_{(\overset{o}{\pi}_i)}$ be the set of arbitrage-free refinements of $(\overset{o}{\pi}_i)$, (partially) ordered by the relation $\leq$. This set, $\mathcal{R}_{(\overset{o}{\pi}_i)}$, is non-empty, for it contains the symmetric self-attainable refinement of $(\overset{o}{\pi}_i)$, along Remark 4. A chain in $\mathcal{R}_{(\overset{o}{\pi}_i)}$ is a totally ordered subset, say $\{(\overset{o}{\pi}{}_i^k)\}_{k \in K}$, where $K$ is a non-empty set, such that for every pair $(k, k') \in K^2$, either $(\overset{o}{\pi}{}_i^k) \leq (\overset{o}{\pi}{}_i^{k'})$ or $(\overset{o}{\pi}{}_i^{k'}) \leq (\overset{o}{\pi}{}_i^k)$. We set as given such a chain, $\{(\overset{o}{\pi}{}_i^k)\}_{k \in K}$. Along Definition 1, we let $\{(P_i^k)\} := \{(P(\overset{o}{\pi}{}_i^k))\}_{k \in K}$ be its chain of supports and, for each $i \in I$, $P_i := \overline{\cup_{k \in K} P_i^k} \subset P(\overset{o}{\pi}_i)$ be a compact set, and $\overset{o}{\overline{\pi}}_i \in \mathcal{CB}$ be the class of beliefs with support $P(\overset{o}{\overline{\pi}}_i) = P_i$. Then, by construction, $(\overset{o}{\pi}{}_i^k) \leq (\overset{o}{\overline{\pi}}_i) \leq (\overset{o}{\pi}_i)$ holds, for every $k \in K$.

Assume, by contraposition, that $(\overset{o}{\overline{\pi}}_i) \notin \mathcal{R}_{(\overset{o}{\pi}_i)}$, i.e., from Claim 2, there exists $(z_i) \in (\mathbb{R}^J)^I$, such that $\sum_{i=1}^m z_i = 0$ and $V(\omega_i) \cdot z_i \geqslant 0$ for every couple $(i, \omega_i) \in I \times P_i$, with at least one strict inequality, say $V(\omega_1) \cdot z_1 > 0$ for $\omega_1 \in P_1$. Since, for all $k \in K$, the class structure $(\overset{o}{\pi}{}_i^k) \in \mathcal{CSB}$ is arbitrage-free and such that $(\overset{o}{\pi}{}_i^k) \leq (\overset{o}{\overline{\pi}}_i)$, the latter relations imply, from Claim 2, that $V(\omega_i^k) \cdot z_i = 0$ for every triple $(k, i, \omega_i^k) \in K \times I \times P_i^k$. Since $V$ is continuous, the relation $V(\omega_1) \cdot z_1 > 0$ for $\omega_1 \in P_1 := \overline{\cup_{k \in K} P_1^k}$ implies that there exists $k \in K$ and $\omega_1^k \in P_1^k$, such that $V(\omega_1^k) \cdot z_1 > 0$, in contradiction with the above equalities. This contradiction shows that $(\overset{o}{\pi}_i) \in \mathcal{R}_{(\overset{o}{\pi}_i)}$, hence, from above, that $(\overset{o}{\pi}_i)$ is an upper bound of the chain $\{(\overset{o}{\pi}{}_i^k)\}_{k\in K}$ in $\mathcal{R}_{(\overset{o}{\pi}_i)}$. Along Zorn's Lemma (Aliprantis-Border, 1999, p. 14), $\mathcal{R}_{(\overset{o}{\pi}_i)}$ has a maximal element, that is, a refinement, $(\overset{o}{\pi}{}_i^*) \leq (\overset{o}{\pi}_i)$, which satisfies Conditions $(i)$-$(ii)$ of Claim 3, and which is unique from the latter Condition $(ii)$. Let $\overset{o}{\Pi}[V, (\overset{o}{\pi}_i)] \in \mathcal{CSB}$ be that maximal element. From above, the set $\mathcal{R}_{(\overset{o}{\pi}_i)}$ contains the symmetric self-attainable refinement, which refines $\overset{o}{\Pi}[V, (\overset{o}{\pi}_i)]$, from Condition $(ii)$. Hence, $\overset{o}{\Pi}[V, (\overset{o}{\pi}_i)]$ is self-attainable. This completes the proof. $\square$

We now study how prices may convey information in this model.

**Claim 4** *Let a class structure of payoffs and beliefs,* $[(V, (\overset{o}{\pi}_i)]$, *and a price,* $q \in \mathbb{R}^J$, *be given. Then, for each* $i \in I$, *there exists a set,* $\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q) \in \varnothing \cup \mathcal{CB}$, *such that:*
$(i)$ *either* $\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q) = \varnothing$, *or* $\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q) \leq \overset{o}{\pi}_i$ *along Definition 1;*
$(ii)$ *if* $\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q) \leq \overset{o}{\pi}_i$, *then,* $(V, \overset{o}{\Pi}(V, \overset{o}{\pi}_i, q))$ *is* $q$*-arbitrage-free along Definition 3;*
$(iii)$ *every* $q$*-arbitrage-free refinement of* $\overset{o}{\pi}_i$ *refines* $\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q)$.
*Moreover, for every pair* $(i, \pi_i^*) \in I \times \overset{o}{\Pi}(V, \overset{o}{\pi}_i, q)$, *there exists a continuous mapping* $\lambda_i : P(\pi_i^*) \to \mathbb{R}_{++}$, *such that* $q = \int_{\omega \in P(\pi_i^*)} \lambda_i(\omega) V(\omega) d\pi_i^*(\omega)$.

**Proof** The proof is similar to that of Claim 3. Let $i \in I$, $q \in \mathbb{R}^J$, and a class structure, $(\overset{o}{\pi}_i) \in \mathcal{CSB}$, along Definition 1, be given. Let $\mathcal{R}_{(\overset{o}{\pi}_i, q)}$ be the set of $q$-arbitrage-free refinements of $\overset{o}{\pi}_i \in \mathcal{CB}$, (partially) ordered by the relation $\leq$ of Definition 1. Then, either $\mathcal{R}_{(\overset{o}{\pi}_i, q)} = \varnothing$, and we let $\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q) = \varnothing$, or $\mathcal{R}_{(\overset{o}{\pi}_i, q)} \neq \varnothing$, which we henceforth assume. Along Definition 1, let $\{\overset{o}{\pi}{}_i^k\}_{k\in K}$ be a chain in $\mathcal{R}_{(\overset{o}{\pi}_i, q)}$, $\{P_i^k\} := \{P(\overset{o}{\pi}{}_i^k)\}_{k\in K}$ be the related chain of supports, $P_i := \overline{\cup_{k\in K} P_i^k} \subset P(\overset{o}{\pi}_i)$ be compact, and $\overset{\overset{o}{=}}{\pi}_i \in \mathcal{CB}$ be the class of beliefs with support $P(\overset{\overset{o}{=}}{\pi}_i) = P_i$. By construction, $\overset{o}{\pi}{}_i^k \leq \overset{\overset{o}{=}}{\pi}_i \leq \overset{o}{\pi}_i$, for all $k \in K$.

Assume, by contraposition, that $\overset{\overset{o}{=}}{\pi}_i \notin \mathcal{R}_{(\overset{o}{\pi}_i, q)}$, i.e., from Definition 3, there exists $z \in \mathbb{R}^J$, such that $-q \cdot z \geqslant 0$ and $V(\omega) \cdot z \geqslant 0$ for every $\omega \in P_i$, with at least one strict inequality, say $(V(\overline{\omega}) \cdot z - q \cdot z) > 0$ for $\overline{\omega} \in P_i$. Since, $\overset{o}{\pi}{}_i^k$ is $q$-arbitrage-free and $P_i^k \subset P_i$, for every $k \in K$, the above relations imply, from Definition 3, that $-q \cdot z = 0$ and $V(\omega^k) \cdot z = 0$, for every $(k, \omega^k) \in K \times P_i^k$. Since $V$ is continuous, the relation $V(\overline{\omega}) \cdot z > 0$, which holds for $\overline{\omega} \in P_i := \overline{\cup_{k \in K} P_i^k}$, implies that there exists $k \in K$ and $\omega_i^k \in P_i^k$, such that $V(\omega_i^k) \cdot z > 0$, in contradiction with the above equalities. This contradiction yields $\overset{o}{\pi}_i \in \mathcal{R}_{(\overset{o}{\pi}_i, q)}$, hence, from above, $\overset{o}{\pi}_i$ is an upper bound of the chain $\{\overset{o}{\pi}{}_i^k\}_{k \in K}$ in $\mathcal{R}_{(\overset{o}{\pi}_i, q)}$. From Zorn's Lemma (Aliprantis-Border, 1999, p. 14), $\mathcal{R}_{(\overset{o}{\pi}_i, q)}$ has a maximal element, i.e., a refinement, $\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q) \leq \overset{o}{\pi}_i$, which meets Conditions $(ii)$ and $(iii)$ of Claim 4. The last part of Claim 4 results from Definition 3 and from above. □

**Definition 4** *Let a class structure of payoffs and beliefs,* $[V, (\overset{o}{\pi}_i)]$, *and a price,* $q \in \mathbb{R}^J$, *be given. The information (set) revealed by price* $q$ *to the generic* $i^{th}$ *agent is the (possibly empty) set,* $\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q) \in \varnothing \cup \mathcal{CB}$, *of Claim 4, also denoted by* $\overset{o}{\Pi}(\overset{o}{\pi}_i, q)$.

### 3.3 Structures of beliefs revealed by prices

The following Claim characterizes no-arbitrage prices along Definition 3.

**Claim 5** *Let a price,* $q \in \mathbb{R}^J$, *a class structure of payoffs and beliefs,* $[V, (\overset{o}{\pi}_i)]$, *and the set collection,* $(\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q)) := (\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q))_{i \in I}$, *along Definition 4, be given. Then, the following statements are equivalent:*

$(i)$ $q$ *is a no-arbitrage price of* $[V, (\overset{o}{\pi}_i)]$, *i.e.,* $q \in Q[V, (\overset{o}{\pi}_i)]$;

$(ii)$ $(\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q))$ *is the coarsest* $q$*-arbitrage-free refinement of* $(\overset{o}{\pi}_i)$;

$(iii)$ $(\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q)) \leq (\overset{o}{\pi}_i)$, *i.e.,* $(\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q))$ *is a refinement of* $(\overset{o}{\pi}_i)$;

$(iv)$ $\cap_{i=1}^m P(\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q)) \neq \varnothing$, *i.e.,* $(\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q))$ *is a class structure of beliefs.*

**Proof** $(i) \Rightarrow (ii)$ Let $q \in Q[V, (\overset{o}{\pi}_i)]$ be given. Then, along Definition 3, there exists a $q$-arbitrage-free refinement, $(\overset{o}{\pi}{}_i^*)$, of $(\overset{o}{\pi}_i)$, which we set as given, and, for each $i \in I$, the set, $\mathcal{R}_{(\overset{o}{\pi}_i, q)}$, of $q$-arbitrage-free refinements of $\overset{o}{\pi}_i$ is non-empty. Resuming exactly the proof of Claim 4, at this stage ($\mathcal{R}_{(\overset{o}{\pi}_i,q)} \neq \varnothing$) and in all subsequent steps, we let the reader check that the set $\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q)$ (of Definition 4) is non-empty and meets Conditions $(i)$-$(ii)$-$(iii)$ of Claim 4. The latter Conditions imply that, for each $i \in I$, $\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q)$ is the unique coarsest $q$-arbitrage-free refinement of $\overset{o}{\pi}_i$, hence, from above, that $\overset{o}{\pi}_i^* \leq \overset{o}{\Pi}(V, \overset{o}{\pi}_i, q) \leq \overset{o}{\pi}_i$. Since $(\overset{o}{\pi}_i^*) \in \mathcal{CSB}$, the latter assertions, which hold and are written for each $i \in I$, imply that the collection $(\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q))$ is a refinement of $(\overset{o}{\pi}_i)$ and, moreover, the (unique) coarsest $q$-arbitrage-free refinement of $(\overset{o}{\pi}_i)$.

The relations $(ii) \Rightarrow (iii) \Rightarrow (iv)$ are immediate from Definition 1.

$(iv) \Rightarrow (i)$ If $(\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q)) \in \mathcal{CSB}$, then, $(\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q))$ refines $(\overset{o}{\pi}_i)$, from Claim 4-$(i)$, and is $q$-arbitrage-free, from Claim 4-$(ii)$, that is, $q \in Q_c[V, (\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q))] \subset Q[V, (\overset{o}{\pi}_i)]$. □

**Definition 5** *Let a class structure of payoffs and beliefs, $[V, (\overset{o}{\pi}_i)]$, and no-arbitrage price, $q$, be given. The coarsest $q$-arbitrage-free refinement of $(\overset{o}{\pi}_i)$ along Claim 5, $(\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q))$, also denoted by $(\overset{o}{\Pi}(\overset{o}{\pi}_i, q))$, is said to be revealed by price $q \in Q[V, (\overset{o}{\pi}_i)]$. A refinement, $(\overset{o}{\pi}_i^*) \leq (\overset{o}{\pi}_i)$ (or, equivalently, $(\pi_i^*) \leq (\pi_i) \in \Pi_{i=1}^m \overset{o}{\pi}_i$) is said to be price-revealable, if there exists $\overline{q} \in \mathbb{R}^J$, such that $(\overset{o}{\pi}_i^*) = (\overset{o}{\Pi}(V, \overset{o}{\pi}_i, \overline{q}))$ (or $(\pi_i^*) \in \Pi_{i=1}^m \overset{o}{\Pi}(V, \overset{o}{\pi}_i, \overline{q})$).*

*Remark 5* As shown in Cornet-de Boisdeffre (2002), given $((\overset{o}{\pi}_i), q) \in \mathcal{CSB} \times Q[(\overset{o}{\pi}_i)]$, there may exist other $q$-arbitrage-free refinements of $(\overset{o}{\pi}_i)$ than the coasest, $(\overset{o}{\Pi}(\overset{o}{\pi}_i, q))$.

Claim 6 shows that the coarsest arbitrage-free refinement is always price-revealable.

**Claim 6** *Let $(\overset{o}{\pi}_i) \in \mathcal{CSB}$ be given. Along Claim 3 and Definition 5, let $\overset{o}{\Pi}[(\overset{o}{\pi}_i)]$ and $(\overset{o}{\Pi}(\overset{o}{\pi}_i, q))$, for every $q \in Q[V, (\overset{o}{\pi}_i)]$, be, respectively, the coarsest arbitrage-free, and coarsest $q$-arbitrage-free, refinements of $(\overset{o}{\pi}_i)$. Then, the following assertion holds:*
*$(i)$ $\varnothing \neq \{q \in \mathbb{R}^J : \overset{o}{\Pi}[(\overset{o}{\pi}_i)] = (\overset{o}{\Pi}(\overset{o}{\pi}_i, q))\} = Q_c[V, \overset{o}{\Pi}[(\overset{o}{\pi}_i)]]$.*
*Hence, $\overset{o}{\Pi}[(\overset{o}{\pi}_i)]$ can be revealed by any price $q \in Q_c[V, \overset{o}{\Pi}[(\overset{o}{\pi}_i)]]$. Moreover, if $[V, (\overset{o}{\pi}_i)]$ is arbitrage-free, $\overset{o}{\Pi}[(\overset{o}{\pi}_i)] = (\overset{o}{\pi}_i)$ and $(\overset{o}{\pi}_i)$ can be revealed by any price $q \in Q_c[V, (\overset{o}{\pi}_i)] \neq \varnothing$.*

**Proof** Let $(\overset{o}{\pi}_i) \in \mathcal{CSB}$ be given and let $\overset{o}{\Pi}[(\overset{o}{\pi}_i)]$ and $(\overset{o}{\Pi}(\overset{o}{\pi}_i, q))$, for every $q \in Q[V, (\overset{o}{\pi}_i)]$, be defined as in Claim 6. From Claim 3 and Definition 3, the set $Q_c[V, \overset{o}{\Pi}[(\overset{o}{\pi}_i)]]$ is non-empty. Let $q \in Q_c[V, \overset{o}{\Pi}[(\overset{o}{\pi}_i)]]$ be given. Then, from Claim 5, the relation $\overset{o}{\Pi}[(\overset{o}{\pi}_i)] \leq (\overset{o}{\Pi}(\overset{o}{\pi}_i, q))$ holds, since $\overset{o}{\Pi}[(\overset{o}{\pi}_i)$ is $q$-arbitrage-free, whereas, from Claim 3, the converse relation, $(\overset{o}{\Pi}(\overset{o}{\pi}_i, q)) \leq \overset{o}{\Pi}[(\overset{o}{\pi}_i)]$, also holds, since $\overset{o}{\Pi}[(\overset{o}{\pi}_i)$ is the coarsest arbitrage-free refinement. Hence, $\overset{o}{\Pi}[(\overset{o}{\pi}_i) = (\overset{o}{\Pi}(\overset{o}{\pi}_i, q)) \leq (\overset{o}{\pi}_i)$. We have thus shown the relations: $\varnothing \neq Q_c[V, \overset{o}{\Pi}[(\overset{o}{\pi}_i)]] \subset \{q \in \mathbb{R}^J : \overset{o}{\Pi}[(\overset{o}{\pi}_i)] = (\overset{o}{\Pi}(\overset{o}{\pi}_i, q))\}$. Conversely, let $q \in \mathbb{R}^J$ be given, such that $\Pi[(\overset{o}{\pi}_i)] = (\overset{o}{\Pi}(\overset{o}{\pi}_i, q))$. Then, from Claim 5, $\Pi[(\overset{o}{\pi}_i)]$ is $q$-arbitrage-free, i.e., $q \in Q_c[V, \overset{o}{\Pi}[(\overset{o}{\pi}_i)]]$. Hence, $\{q \in \mathbb{R}^J : \overset{o}{\Pi}[(\overset{o}{\pi}_i)] = (\overset{o}{\Pi}(\overset{o}{\pi}_i, q))\} \subset Q_c[V, \overset{o}{\Pi}[(\overset{o}{\pi}_i)]]$ and, from above, assertion $(i)$ holds. The rest is immediate from the definitions and assertion $(i)$. $\square$

We now characterize prices, which reveal a self-attainable refinement.

**Claim 7** *Let $(\overset{o}{\pi}_i) \in \mathcal{CSB}$ & $q \in \mathbb{R}^J$ be given. Let $\mathcal{AS}$ be the non-empty set of arbitrage-free self-attainable refinements of $(\overset{o}{\pi}_i)$. The following assertions are equivalent:*

$(i)$ $q \in \cup_{(\overset{o}{\pi}{}^*_i) \in \mathcal{AS}} Q_c[V, (\overset{o}{\pi}{}^*_i)]$, *i.e., $q$ is a self-attainable no-arbitrage price;*

$(ii)$ $(\overset{o}{\Pi}(\overset{o}{\pi}_i, q))$ *is a self-attainable refinement of $(\overset{o}{\pi}_i)$.*

**Proof** From Claim 3 and Remark 4, the set $\mathcal{AS}$ contains both the symmetric self-attainable and the coarsest arbitrage-free refinements of $(\overset{o}{\pi}_i)$. The relation $(i) \Rightarrow (ii)$ is a direct consequence of Claim 5. Moreover, if assertion $(ii)$ holds, $(\overset{o}{\Pi}(\overset{o}{\pi}_i, q)) \leq (\overset{o}{\pi}_i)$ is $q$-arbitrage-free, from Claim 5, i.e., $q \in Q_c[V, (\overset{o}{\Pi}(\overset{o}{\pi}_i, q))]$, and assertion $(i)$ holds. $\square$

We now examine how agents, endowed with no price model a la Radner, may still update their beliefs from observing markets, namely, prices, if they can meet agreement, or exchange opportunities, if this is not the case.

## 4 Inferring information from prices or trade opportunities

This Section generalizes Cornet-de Boisdeffre (2009) to the above economy $\mathcal{E}$.

### 4.1 Sequential refinement through prices

Throughout, we let a class structure, $[V, (\overset{o}{\pi}_i)]$, of payoffs and beliefs, an agent, $i \in I$, and an asset price, $q \in \mathbb{R}^J$, be given. We study how the generic $i^{th}$ agent, endowed with the initial expectation set, $P(\overset{o}{\pi}_i)$, along Definition 1, may narrow down in steps her expectation set from observing the current price, $q$, on financial markets, without having any model of how market prices are determined. To that aim, we define, by induction on $n \in \mathbb{N}$, two sequences, $\{A_i^n\}_{n\in\mathbb{N}}$ and $\{P_i^n\}_{n\in\mathbb{N}}$, of sub-sets of $\varnothing \cup \mathcal{M}$, called, respectively, the arbitrage and (interim) expectation sets:

- for $n = 0$, we let $A_i^0 = \varnothing$ and $P_i^0 := P(\overset{o}{\pi}_i)$;
- for $n \in \mathbb{N}$ arbitrary, with $A_i^n$ and $P_i^n$ defined at step $n$, we let $A_i^{n+1} := P_i^{n+1} := \varnothing$, if $P_i^n = \varnothing$, and, otherwise:

$A_i^{n+1} := \{\overline{\omega} \in P_i^n : \exists z \in \mathbb{R}^J,\ -q \cdot z \geqslant 0,\ V(\overline{\omega}) \cdot z > 0 \textit{ and } V(\omega) \cdot z \geqslant 0,\ \forall \omega \in P_i^n\}$;

$P_i^{n+1} := P_i^n \setminus A_i^{n+1}$, i.e., the agent rules out expectations, granting an arbitrage.

**Claim 8** *Let $(\overset{o}{\pi}_i) \in \mathcal{CSB}$ and $q \in \mathbb{R}^J$ be given. Let $\overset{o}{\Pi}(\overset{o}{\pi}_i, q)$ be the set of Definition 4 and $\{A_i^n\}_{n\in\mathbb{N}}$ and $\{P_i^n\}_{n\in\mathbb{N}}$, be defined from above. Let $P(\overset{o}{\Pi}(\overset{o}{\pi}_i, q))$ be empty, if $\overset{o}{\Pi}(\overset{o}{\pi}_i, q)$ is empty, and the support of $\overset{o}{\Pi}(\overset{o}{\pi}_i, q) \in \mathcal{CB}$ otherwise. The following assertions hold:*

*(i) $\exists N \in \mathbb{N} : \forall n > N,\ A_i^n = \varnothing$ and $P_i^n = P_i^N$;*

*(ii) $P_i^N = \lim\ P_i^n = P(\overset{o}{\Pi}(\overset{o}{\pi}_i, q))$, along assertion (i).*

**Proof** Let $(\overset{o}{\pi}_i) \in \mathcal{CSB}$ and $q \in \mathbb{R}^J$ be given, and let $\overset{o}{\Pi}(\overset{o}{\pi}_i, q)$, $P(\overset{o}{\Pi}(\overset{o}{\pi}_i, q))$, $\{A_i^n\}_{n\in\mathbb{N}}$ and $\{P_i^n\}_{n\in\mathbb{N}}$ be defined as in Claim 8 and denote $P_i^* := \cap_{i=1}^m P_i^n = \lim_{n\to\infty} \searrow P_i^n$.

With a non-restrictive convention that the empty set is included in any other set, we show, first, by induction on $n \in \mathbb{N}$, that $P(\overset{o}{\Pi}(\overset{o}{\pi}_i, q)) \subset P_i^n$, for every $n \in \mathbb{N}$. Indeed, from Claim 4, the relation $P(\overset{o}{\Pi}(\overset{o}{\pi}_i, q)) \subset P_i^0$ holds for $n = 0$. Assume, now, by contraposition, that, for some $n \in \mathbb{N}$, $P(\overset{o}{\Pi}(\overset{o}{\pi}_i, q)) \subset P_i^n$ and $P(\overset{o}{\Pi}(\overset{o}{\pi}_i, q)) \not\subset P_i^{n+1}$. Then, there exist $\overline{\omega} \in \ P(\overset{o}{\Pi}(\overset{o}{\pi}_i, q)) \ \cap \ A_i^{n+1}$ and $z \in \mathbb{R}^J$, such that $-q \cdot z \geqslant 0$, $V(\overline{\omega}) \cdot z > 0$ and $V(\omega) \cdot z \geqslant 0$, for every $\omega \in P(\overset{o}{\Pi}(\overset{o}{\pi}_i, q)) \subset P_i^n$, which contradicts Claim 5, along which $(V, P(\overset{o}{\Pi}(\overset{o}{\pi}_i, q)))$ is $q$-arbitrage-free, if non-empty.

Hence, the relation $P(\overset{o}{\Pi}(\overset{o}{\pi}_i, q)) \subset P_i^n$ holds for every $n \in \mathbb{N}$.

Assume, first, that $P_i^* := \cap_{n \in \mathbb{N}} P_i^n = \varnothing$. Since the sequence $\{P_i^n\}_{n \in \mathbb{N}}$ is non-increasing and made of compact or empty sets, there exists $N \in \mathbb{N}$, such that $P_i^n = A_i^n = \varnothing$, for every $n \geqslant N$, and, from above, assertions $(i)$-$(ii)$ of Claim 8 hold.

Assume, next, that $P_i^* \neq \varnothing$, then, $P_i^*$, a non-empty intersection of compact sets, is compact, and, from above, $P(\overset{o}{\Pi}(\overset{o}{\pi}_i, q)) \subset P_i^*$. Along Definition 1, we let $\overset{o}{\pi}{}_i^* \in \mathcal{CB}$ be the (unique) refinement of $\overset{o}{\pi}_i$, defined by $P(\overset{o}{\pi}{}_i^*) = P_i^*$, and consider two cases.

First, we assume that assertion $(i)$ of Claim 8 holds and let $N \in \mathbb{N}$ be such that $A_i^{N+1} = \varnothing$. Then, by construction, $P_i^N = P_i^*$, and $\overset{o}{\pi}{}_i^* \leq \ \overset{o}{\pi}_i$ is $q$-arbitrage-free (since $A_i^{N+1} = \varnothing$), which implies, from Claim 4, $\overset{o}{\pi}{}_i^* \leq \ \overset{o}{\Pi}(\overset{o}{\pi}_i, q)$. The latter relation yields $P(\overset{o}{\pi}{}_i^*) := P_i^* \subset P(\overset{o}{\Pi}(\overset{o}{\pi}_i, q))$, and, from above, assertions $(i)$-$(ii)$ of Claim 8 hold again.

Second, we assume, by contraposition, that assertion $(i)$ of Claim 8 fails, that is:

$\forall n \in \mathbb{N}, \ \exists (\omega_n, z_n) \in P_i^n \times Z_i^n : \ -q \cdot z_n \geqslant 0, \ V(\omega_n) \cdot z_n > 0 \ \textit{and} \ V(\omega) \cdot z_n \geqslant 0, \ \forall \omega \in P_i^n,$

in which we may take $Z_i^n := \{z \in \mathbb{R}^J : \|z\| = 1, \ z \in \langle Z_i^{on} \rangle^{\perp}\}$ orthogonal to

$Z_i^{on} := \{z \in \mathbb{R}^J : -q \cdot z = 0, \ V(\omega) \cdot z = 0, \ \forall \omega \in P_i^n\}.$

Since $\{P_i^n\}_{n\in\mathbb{N}}$ is decreasing, the sequence of vector spaces, $\{Z_i^{on}\}$, is non-decreasing in $\mathbb{R}^J$, hence, stationary. We let $N \in \mathbb{N}$ be such that $Z_i^n = Z_i^N$, for every $n \geqslant N$, and denote simply $Z_i := Z_i^N$ and $Z_i^o := Z_i^{oN}$. Since $P_i^0 \times Z_i$ is compact, we may assume there exists $(\omega_*, z_*) = \lim_{n\to\infty}(\omega_n, z_n) \in P_i^0 \times Z_i$. We recall that $Z_i$ and $Z_i^o$ are orthogonal and notice that $Z_i^o = \{z \in \mathbb{R}^J : -q \cdot z = 0,\ V(\omega)\cdot z = 0,\ \forall \omega \in P_i^*\}$.

The above relations on $(\omega_n, z_n)$, for each $n \in \mathbb{N}$, and the continuity of $V$ and of the scalar product imply that $-q \cdot z_* \geqslant 0$ and $V(\omega)\cdot z_* \geqslant 0$, for every $\omega \in P_i^*$, with one strict inequality, since $z_* \in Z_i$, is orthogonal to $Z_i^o$ and such that $\|z_*\| = 1$. Hence, there exists $\omega^* \in P_i^*$, such that $V(\omega^*)\cdot z_* > 0$. By construction, $P_i^* := \cap_{n\in\mathbb{N}} P_i^n$ is disjoint from $A_i^* := \cup_{n\in\mathbb{N}} A_i^n$. Hence, $\omega^* \notin A_i^*$. From the continuity of the scalar product and above, there exists $n > N$, such that $V(\omega^*)\cdot z_n > 0$, which implies, from the above relations on $(\omega_n, z_n)$, that $\omega^* \in A_i^{n+1}$, in contradiction with the fact that $\omega^* \notin A_i^*$. This contradiction proves that assertion $(i)$ of Claim 8 holds, and completes the proof. $\square$

The joint resuls of Claims 4, 5 and 8 yield the following Definition.

**Definition 6** *Given a class structure of payoffs and beliefs,* $[V, (\overset{o}{\pi}_i)]$*, and price,* $q \in \mathbb{R}^J$*, we let, for each* $i \in I$*,* $\{P_i^n\}_{n\in\mathbb{N}}$ *and* $P_i^* := \lim_{n\to\infty} P_i^n$*, be, respectively, the set sequence and the final expectation set, defined above by induction. The information revealed by price* $q$ *to the generic* $i^{th}$ *agent along the no-arbitrage principle is the (possibly empty) set of beliefs, denoted by* $\overset{o}{\Delta}(V, \overset{o}{\pi}_i, q) \in \varnothing \cup \mathcal{CB}$*, whose support is* $P_i^*$*. This set coincides with the information set,* $\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q)$*, revealed by price* $q$ *of Definition 4. If* $q \in \mathbb{R}^J$ *is a no-arbitrage price, the refinement revealed by price* $q$ *along the no-arbitrage principle is the class structure,* $(\overset{o}{\Delta}(V, \overset{o}{\pi}_i, q)) \leq (\overset{o}{\pi}_i)$*, defined from above, which coincides with the refinement,* $(\overset{o}{\Pi}(V, \overset{o}{\pi}_i, q))$*, revealed by price* $q$ *of Definition 5.*

The meaning of the no-arbitrage principle of Definition 6 is now clear. Agents having no clue of how market prices are determined can still update their beliefs from observing current prices so as to rule out arbitrage. They will, in any case, reach their final update after a finite number of inference steps. As long as markets have not reached a no-arbitrage price, traders cannnot agree on prices and a sequential equilibrium may not exist. Claims 5 and 8 above show, jointly, that agents have common expectations - a necessary condition for sequential equilibrium to exist - if, and only if, the asset price is a no-arbitrage price (and such prices exist).

Along the same Claims, once markets have reached a no-arbitrage price, decentralized agents, who update their beliefs from the no-arbitrage principle, may infer a refinement, agree on current prices, and share common forecasts. From Claim 5, 7 and 8, whenever the no-arbitrage price is self-attainable, the refinement inferred is self-attainable. We now show that a similar refinement process is possible without observing such a price. Agents may infer information from trade opportunities on markets, in a way which leads to preclude arbitrage, after finitely many inference steps. Then, agents infer the coarsest arbitrage-free refinement of Claim 3, which may also be revealed, along Claim 6, by any of its common no-arbitrage prices.

### 4.2 Sequential refinement through trade

Throughout, we let a class structure, $[V, (\overset{o}{\pi}_i)]$, be given. We study how agents may narrow down in steps their expectation sets from observing exchange opportunities on financial markets. As above, we define, by induction on $n \in \mathbb{N}$, two sequences, $\{A^n\}_{n\in\mathbb{N}} := \{\Pi_{i=1}^m A_i^n\}_{n\in\mathbb{N}}$ and $\{P^n\}_{n\in\mathbb{N}} := \{\Pi_{i=1}^m P_i^n\}_{n\in\mathbb{N}}$, of sub-sets of $(\varnothing \cup \mathcal{M})^m$:

- we let $A_i^0 = \varnothing$ and $P_i^0 := P(\overset{o}{\pi}_i)$, for each $i \in I$, and $A^0 := \Pi_{i=1}^m A_i^0$ and $P^0 := \Pi_{i=1}^m P_i^0$;
- with $A^n := \Pi_{i=1}^m A_i^n$ and $P^n := \Pi_{i=1}^m P_i^n$ defined at step $n \in \mathbb{N}$, we let, for each $i' \in I$:

$$A_{i'}^{n+1} := \{\overline{\omega} \in P_{i'}^n : \exists (z_i) \in (\mathbb{R}^J)^m,\ \textstyle\sum_{i=1}^m z_i{=}0,\ \text{-}q{\cdot}z_i{\geqslant}0,\ V(\overline{\omega}){\cdot}z_{i'}{>}0,\ V(\omega_i){\cdot}z_i{\geqslant}0,\ \forall (i, \omega_i) \in I{\times}P_i^n\}$$

$P_{i'}^{n+1} := P_{i'}^{n} \setminus A_{i'}^{n+1}$, i.e., agents rule out expectations, granting an arbitrage,

$A^{n+1} := \Pi_{i=1}^{m} A_i^{n+1}$and $P^{n+1} := \Pi_{i=1}^{m} P_i^{n+1}$.

**Claim 10** *Let $(\overset{o}{\pi}_i) \in \mathcal{CSB}$ be given. We will denote simply $(\overset{o}{\pi}{}_i^*) := \overset{o}{\Pi}[(\overset{o}{\pi}_i)] \in \mathcal{CSB}$ the coarsest arbitrage-free refinement of $(\overset{o}{\pi}_i)$ along Claim 3, and let, for each $i \in I$, $P_i^* := P(\overset{o}{\pi}{}_i^*)$ be the support of $\overset{o}{\pi}{}_i^* \in \mathcal{CB}$, and $P^* := \Pi_{i=1}^{m} P_i^*$. Let $\{A^n\}_{n \in \mathbb{N}}$ and $\{P^n\}_{n \in \mathbb{N}}$, be defined from above. Then, the following assertions hold:*

*(i) $\exists N \in \mathbb{N} : \forall n > N,\ A^n = \varnothing$ and $P^n = P^N$;*

*(ii) $P^N = \lim_{n \to \infty}\ P^n = P^*$, along assertion $(i)$.*

**Proof** Let $(\overset{o}{\pi}_i) \in \mathcal{CSB}$ be given, and let $(\overset{o}{\pi}{}_i^*)$, $P^* := \Pi_{i=1}^{m} P_i^* := \Pi_{i=1}^{m} P(\overset{o}{\pi}{}_i^*)$, $\{A^n\}_{n \in \mathbb{N}} := \{\Pi_{i=1}^{m} A_i^n\}_{n \in \mathbb{N}}$ and $\{P^n\}_{n \in \mathbb{N}} := \{\Pi_{i=1}^{m} P_i^n\}_{n \in \mathbb{N}}$ be defined as in Claim 10 and denote $P^{**} := \Pi_{i=1}^{m} P_i^{**} := \cap_{i=1}^{m} P^n = \lim_{n \to \infty} \searrow P^n$.

We show, first, by induction on $n \in \mathbb{N}$, that $P_i^* \subset P_i^n$, for every pair $(i, n) \in I \times \mathbb{N}$. Let $i' \in I$ be given. From Claim 3, the relation $P_{i'}^* \subset P_{i'}^0$ holds for $n = 0$. Assume, now, by contraposition, that, for some $n \in \mathbb{N}$, $P_{i'}^* \subset P_{i'}^n$ and $P_{i'}^* \not\subseteq P_{i'}^{n+1}$. Then, there exist $\overline{\omega} \in\ P_{i'}^*\ \cap\ A_{i'}^{n+1}$ and $(z_i) \in (\mathbb{R}^J)^m$, such that $\sum_{i=1}^{m} z_i = 0$, $V(\overline{\omega}) \cdot z_{i'} > 0$ and $V(\omega_i) \cdot z_i \geqslant 0$, for every $(i, \omega_i) \in I \times P_i^* \subset I \times P_i^n$, which (from Claim 2) contradicts Claim 3, along which $[V, (\overset{o}{\pi}{}_i^*)]$ is arbitrage-free.

Hence, the relation $P_i^* \subset P_i^n$ holds for every pair $(i, n) \in I \times \mathbb{N}$, which implies that $P^{**} := \Pi_{i=1}^{m} P_i^{**} := \cap_{i=1}^{m} P^n$ is a compact set, such that $P^* := \Pi_{i=1}^{m} P_i^* \subset P^{**}$. Let $(\overset{o}{\pi}{}_i^{**}) \in \mathcal{CSB}$ be the refinement of $(\overset{o}{\pi}_i)$, defined by $P(\overset{o}{\pi}{}_i^{**}) = P_i^{**}$, for each $i \in I$. From above, $\overset{o}{\pi}{}_i^*$ refines $\overset{o}{\pi}{}_i^{**}$. We now consider two cases.

First, assume that assertion $(i)$ of Claim 10 holds and let $N \in \mathbb{N}$ satisfy that condition. Then, $P^N = P^{**}$, and the refinement, $(\overset{o}{\pi}{}_i^{**})$, is $q$-arbitrage-free (since $A_i^{N+1} = \varnothing$, for each $i \in I$), which implies, from Claim 3, that $\overset{o}{\pi}{}^{**}_i \leq \overset{o}{\pi}{}^{*}_i$, hence, from above, that $\overset{o}{\pi}{}^{**}_i = \overset{o}{\pi}{}^{*}_i$, that is, assertions $(i)$ and $(ii)$ of Claim 10 hold.

Second, assume, by contraposition, that assertion $(i)$ of Claim 10 fails, that is:

$\forall n \in \mathbb{N}$, $\exists(\omega^n_{i_n}, (z^n_i)) \in P^n_{i_n} \times Z^n : \ V(\omega^n_{i_n}) \cdot z^n_{i_n} > 0 \ and \ V(\omega_i) \cdot z^n_i \geqslant 0$, $\forall (i, \omega_i) \in I \times P^n_i$, where $Z^n := \{(z_i) \in (\mathbb{R}^J)^m : \sum_{i=1}^m z_i \in \sum_{i=1}^m Z^{on}_i, \ \|(z_i)\| = 1, \ (z_i) \in \Pi_{i=1}^m \langle Z^{on}_i \rangle^\perp\}$, in which $\langle Z^{on}_i \rangle^\perp$ is, for each $i \in I$, the orthogonal complement of $Z^{on}_i := \{z \in \mathbb{R}^J : V(\omega) \cdot z = 0, \ \forall \omega \in P^n_i\}$.

Since $\{P^n\}_{n \in \mathbb{N}}$ is decreasing, the sequence of vector spaces, $\{\Pi_{i=1}^m Z^{on}_i\}$, is non-decreasing in $(\mathbb{R}^J)^m$, hence, stationary. We let $N \in \mathbb{N}$ be such that $Z^n = Z^N$, for every $n \geqslant N$, and denote simply $Z := Z^N$ and $Z^o := \Pi_{i=1}^m Z^{oN}_i$. To simplify, we will assume, non restrictively, that $i_n = 1$, for every $n \in \mathbb{N}$. Since $P^0_1 \times Z$ is compact, we may assume there exists $(\omega^*, (z^*_i)) = \lim_{n \to \infty}(\omega^n_1, (z^n_i)) \in P^0_1 \times Z$. We recall that $Z$ and $Z^o$ are orthogonal and notice that $Z^o = \Pi_{i=1}^m \{z \in \mathbb{R}^J : V(\omega) \cdot z = 0, \ \forall \omega \in P^{**}_i\}$.

The above relations on $(\omega^n_1, (z^n_i))$, for each $n \in \mathbb{N}$, and the continuity of $V$ and of the scalar product imply that $V(\omega) \cdot z^*_i \geqslant 0$, for every pair $(i, \omega) \in I \times P^{**}_i$, with one strict inequality, since $(z^*_i) \in Z$, is orthogonal to $Z^o$ and such that $\|(z^*_i)\| = 1$. Hence, there exists $(i, \omega^{**}) \in I \times P^{**}_i$, such that $V(\omega^{**}) \cdot z^*_i > 0$. By construction, $P^{**}_i := \cap_{n \in \mathbb{N}} P^n_i$ is disjoint from $A^{**}_i := \cup_{n \in \mathbb{N}} A^n_i$. Hence, $\omega^{**} \notin A^{**}_i$.

From the definition of $\sum_{i=1}^m Z^{on}_i$ and above, for each $n \in \mathbb{N}$, there exist (attainable) portfolios, $(\overline{z}^n_i) \in \mathbb{R}^{JI}$, such that $\sum_{i=1}^m \overline{z}_i = 0$ and $V(\omega_i) \cdot \overline{z}^n_i = V(\omega_i) \cdot z^n_i \geqslant 0$, for every $(i, \omega_i) \in I \times P^n_i$. From the continuity of the scalar product, the above relations, $V(\omega^{**}) \cdot z^*_i > 0$ and $z^*_i = \lim z^n_i$, imply, for $n \in \mathbb{N}$ large enough, $V(\omega^{**}) \cdot z^n_i = V(\omega^{**}) \cdot \overline{z}^n_i > 0$, with $\omega^{**} \in P^n_i$, hence, $\omega^{**} \in A^{n+1}_i$, from the latter relations on $(\overline{z}^n_i)$. This contradicts the fact that $\omega^{**} \notin A^{**}_i$. Hence, assertion $(i)$ holds, which completes the proof. $\square$

We have now extended to the infinite dimensional economy, $\mathcal{E}$, the basic arbitrage properties of the Cornet-de Boisdeffre model (2002-2009). Proving the extended properties was a perequisite to study existence of the C.F.E. in the general setting, where agents are, both, asymmetrically informed about the future, and endowed with private idiosyncratic beliefs on forthcoming prices.

In the traditional perfect foresight model, we are used to deal with locally isolated predictable equilibria, whose price is common knowledge. When consumers are prone to uncertainty between several private forecasts, a single prediction is no longer possible. Then, the set of sequential equilibria is easily shown to be typically uncountable, with different equilibria related to different possible private beliefs.

Such an indeterminacy leads to define a minimum uncertainty set, as the set of all prices, which might prevail as a sequential equilibrium price tomorrow, for some structure of beliefs today. When beliefs are private or prone to change, any such price is, a priori, possible. We already know that this set is non-empty in a standard economy with purely financial markets, since this economy admits a perfect foresigth (possibly price-revealed) equilibrium (see De Boisdeffre, 2007). In a companion paper, we will extend this result to all types of financial markets.

On the one hand, it is clear that agents endowed with private beliefs need include the minimum uncertainty set into their expectation sets, if their forecasts are to be correct (along Definition 2 above). On the other hand, we will show a converse result. Namely, that a standard economy, where agents' expectations embed that set, always admits a correct foresight (possibly price-revealed) equilibrium, whatever the structure of payoffs and beliefs of the economy. In particular, a correct foresight equilibrium exists in the standard cases, following Hart (1974) and Radner (1979), where the perfect foresight and the rational expectation equilibria fail to exist.

## References

[1] Balasko, Y., 2003: Temporary financial equilibrium, Economic Theory 21 (1).

[2] Cornet, B., De Boisdeffre, L., 2002: Arbitrage and price revelation with asymmetric information and incomplete markets. J. Math. Econ. 38, 393-410.

[3] Cornet, B., De Boisdeffre, L., 2009: Elimination of arbitrage states in asymmetric information models, Econ. Theory 38, 287-293.

[4] De Boisdeffre, L. 2007: No-arbitrage equilibria with differential information: an existence proof, Economic Theory 31, 255-269.

[5] Grandmont, J.-M., 1977: Temporary general equilibrium theory, Econometrica 45 (3), 535-572.

[6] Grandmont, J.-M., 1982: Temporary general equilibrium theory, Handbook of math. econ., North-Holland, 879-922.

[7] Green, J.R., 1973: Temporary general equilibrium in a sequential trading model with spot and futures transactions, Econometrica 41, 1103-1123.

[8] Hammond, P.J., 1983: Overlapping expectations and Hart's condition for equilibrium in a securities model, JET 31 (1), 170-175.

[9] Hart, O., 1974: On the existence of equilibrium in a securities model, JET 9, 293-311.

[10] Hicks, J.R., 1939: Value and Capital, Clarendon Press, Oxford.

[11] Radner, R., 1972: Existence of equilibrium plans, prices and price expectations in a sequence of markets. Econometrica 40, 289-303.

[12] Radner, R., 1979: Rational expectations equilibrium: generic existence and the information revealed by prices. Econometrica 47, 655-678.